Solution of Second Order Ordinary Differential Equation with Periodic Solutions

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Abstract
In this paper, a discrete Implicit Linear Multistep Method (LMM) of Direct Solution of IVP second order ordinary differential equation with periodic solution was developed at step length k = 3, using trigonometric function as a basis function. The computational burden and computer time wastage involved in the usual reduction of second order problem into system of first order equations are avoided by this method. The development of the method adopts Taylor series expansion techniques and Boundary Locus stability test method. The developed method was found to be accurate, consistent, zero stable, P-stable and convergent. The method was used to solve sample problem on second order ordinary differential equation with periodic solutions and results are quite suitable when compared with other existing methods.
Key Words; *Oscillatory, Periodic, LMM, ODE, Stability, Trigonometry, Taylor series*

1.0.0 INTRODUCTION

Any function of the form

$$y'' = f(t, y) \qquad a \le t \le b,\ y(a) = y_0,\ y'(a) = \alpha \tag{1.0.1}$$

where f(t + T, y) = f(t, y) where T is the period.

is called initial value problems of second order ordinary differential equation with periodic solution. Solutions to the IVP of the type (1.0.1) are highly oscillatory in nature and thus severely restrict the conventional linear multistep method of such system which often occurs in mechanical system without dissipation, satellite tracking and celestial mechanics [[7], [9], [3]]

One of the conditions that such equation (1.0.1) must be satisfied in order to ensure the existence and uniqueness of solution is contained in theorem postulated by [5].

According to [7], [8], [12], and [10] ; the commonest method of solving a second order ordinary differential equation of the form (1.0.1) is by reduction of the problem into first ordinary differential equation.

However, a more serious drawback to such technique arises when the given system of equations to be solved cannot be solved explicitly for the derivatives of the highest order and, thereby; become inefficient, uneconomical for a general purpose application.

In this work, a discrete Linear Multistep Method of the form

$$y_{n+k} = \sum_{j=0}^{k-1} a_j y_{n+j} + (h)^2 \sum_{j=0}^{\mathrm{k}-1} \beta_j y''_{n+j} \tag{1.0.2}$$

is developed at step length K = 3 ; for direct solution of second order initial value problems of ordinary differential equation of the form (1.0.1) using trigonometric function as a basis function

2.0 DERIVATION OF THE METHODS

The development of the numerical methods for solution of periodic initial value problems of ordinary differential equation of the form

$y'' = f(t, y) \quad a \le t \le b,\ y(a) = y_0,\ y'(a) = \alpha$ 2.0.0

where f(t + T, y) = f(t, y) where T is the period.

Assuming the theoretical solution of the equation (2.0.0) is of the form

$y(t) = a \cos wt + b \sin wt$ 2.0.1

At, $t = t_n$

$y_n = y(t_n) = a \cos wt_n + b \sin wt_n$ 2.0.2

$y^1(t_n) = y^1{}_n = a\, w \sin wt_n + bw \cos wt_n$

$y^{11}(t_n) = y^{11}{}_n = -w^2 (a \cos wt_n + b \sin wt_n) \quad = f_n$ 2.0.3

$f_n = -w^2 y_n$ 2.0.4

$t_{n+k} = t_n + kh$ where $k = 0, 1, 2, 3, 4.$ and $h = t_{n+1} - t_n$ 2.0.5

Similarly,

At $t = t_{n+1}$

$y(t_{n+1}) = y_{n+1} = a \cos wt_{n+1} + b \sin wt_{n+1}$ 2.0.6

$y^1(t_{n+1}) = y^1{}_{n+1} = awsin\ wt_{n+1} + bwcos\ wt_{n+1}$ 2.0.7

$y^{11}(t_{n+1}) = y^{11}{}_{n+1} = -w^2 (a \cos wt_{n+1} + b \sin wt_{n+1})$

$f_{n+1} = -w^2 y_{n+1}$ 2.0.8

At $t = t_{n+2}$

$y(t_{n+2}) = y_{n+2} = a \cos wt_{n+2} + b \sin wt_{n+2}$ 2.0.9

$y^1(t_{n+2}) = y^1{}_{n+2} = w\, a \sin wt_{n+2} + bw \cos wt_{n+2}$ 2.0.10

$y^{11}(t_{n+2}) = y^{11}{}_{n+2} = -w^2 (a \cos wt_{n+2} + b \sin wt_{n+2}) = f_{n+2}$ 2.0.11

$f_{n+2} = -w^2 y_{n+2}$ 2.0.12

At $t = t_{n+3}$

$y(t) = y(t_{n+3}) = y_{n+3} = a \cos wt_{n+3} + b \sin wt_{n+3}$ 2.0.13

$y^1(t_{n+3}) = y^1(t_{n+3}) = y^1{}_{n+3} = w\, a \sin wt_{n+3} + bw \cos wt_{n+3}$ 2.0.14

$y^{11}(t_{n+3}) = y^{11}(t_{n+3}) = y^{11}{}_{n+3} = w^2 (a \cos wt_{n+3} + b \sin wt_{n+3}) = f_{n+3}$

2.0.15

$f_{n+3} = -w^2 y_{n+3}$ 2.0.16

2.1 METHODOLOGY

Subtracting equation (2.0.9) from equation (2.0.13) to get

$y_{n+3} - y_{n+2} = a(\cos wt_{n+3} - \cos wt_{n+2}) + b(\sin wt_{n+3} - \sin wt_{n+2})$ 2.1.1

By adopting trigonometric difference equation method and simplifying to obtain

$$y_{n+3} - y_{n+2} = -2 \sin \frac{wh}{2}\left[a \sin \frac{w}{2}(2t_n + 5h) + b \cos \frac{w}{2}(2t_n + 5h)\right] \quad 2.1.2$$

Subtract equation (2.0.15) from (2.1.2) to obtain;

$$y_{n+3} - 2y_{n+2} + y_{n+1} = -4 \sin^2 \frac{wh}{2}\left[a \cos w(t_n + 2h) + b \sin w(t_n + 2h)\right] \quad 2.1.3$$

Subtract 2. 0.16 from 2.1.3 to obtain:

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = -8\sin^3\left(\frac{wh}{2}\right)\left[a \cos \frac{w}{2}(2t_n + 3h) + b \sin \frac{w}{2}(2t_n + 3h)\right]$$

2.1.4

Add equation (2.0.3) and (2.0.8)

$$f_{n+1} + f_n = -2w^2 \cos\frac{wh}{2}\left[a\cos\frac{\omega}{2}(2t_n+3h) + b\sin\frac{\omega}{2}(2t_n+3h)\right] \quad 2.1.5$$

Add equations (2.0.15) and (2.0.11) to obtain;

$$f_{n+3} + f_{n+2} = -2w^2 \cos\frac{wh}{2}\left[a\cos\frac{w}{2}(2t_n+5h) + b\sin\frac{w}{2}(2t_n+5h)\right] \quad 2.1.6$$

Similarly adding equation 2.1.5 to 2.1.6 and simplify to get

$$f_{n+3} + 2f_{n+2} + f_{n+1} = -4w^2\cos^2\frac{wh}{2}\left[a\cos w(t_n+2h) + b\sin w(2t_n+2h)\right] \quad 2.1.7$$

In the same way; add equation 2.1.7 to equation 2.1.5 and simplify to get

$$f_{n+3} + 3f_{n+2} + 3f_{n+1} + f_n = -8w^2\cos^3\frac{wh}{2}\left[a\cos\frac{w}{2}(2t_n+3h) + b\sin\frac{w}{2}(2t_n+3h)\right] \quad 2.1.8$$

Divide equation (2.1.4) by equation (2.1.7) to obtain

$$\frac{y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n}{f_{n+3} + 3f_{n+2} + 3f_{n+1} + f_n} = \frac{1}{w^2}\left(\tan\frac{wh}{2}\right)^3 \quad 2.1.9$$

Using Taylor's series expansion to simplify equation (2.1.9) to obtain

$$\frac{y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n}{f_{n+3} + 3f_{n+2} + 3f_{n+1} + f_n} = \frac{1}{w^2}\left(\frac{1}{2} + \frac{w^2h^2}{24} + \frac{w^4h^4}{240} + \ldots\right)^3 \quad 2.1..10$$

Assuming h is sufficiently small such that wh is also small,
then (2.1.10) modifies into

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^2}{2^3}(f_{n+3} + 3f_{n+2} + 3f_{n+1} + f_n) \quad 2.1.11$$

$$y_{n+3} = 3y_{n+2} - 3y_{n+1} + y_n + \frac{h^2}{8}(f_{n+3} + 3f_{n+2} + 3f_{n+1} + f_n) \quad 2.1.12$$

3.0 Definition

Let $\Pi(r, h) = \rho(r) - h\,\delta(r)$ (3.1.0)
denotes the characteristic polynomial equation of the method where $\rho(r)$ and $\delta(r)$ are called first and second characteristics polynomials respectively; as explained by [11].
In the spirits of [4], [10], [15], [12] [13], [14]; a linear multistep method is said to be consistent if and only if it satisfies the following conditions

$$\left.\begin{array}{l} (i)\ The\ \ order\ \ P \ge 1 \\ (ii)\ \sum_{j=0}^{k} \alpha j = 0 \\ (iii)\ \ \rho(r) = \rho^1(r) = 0 \\ (iv)\ \rho^{11}(r) = 2!\,\delta(r) \end{array}\right\} \quad . \quad (.3.1.1)$$

With the principal root $r \le 1$
from (3.2.1) the characteristic polynomial equation of the 2-step method (1.2.14) is

$$r^3 + 3r^2 + 3r + 1 = \frac{1}{8}(r^3 + 3r^2 + 3r + 1)$$

where ρ(r) = $\rho(r) = r^3 - 3r^2 + 3r - 1$; $and\ \delta(r) = \frac{1}{8}(r^3 + 3r^2 + 3r + 1)$

Simplifying equation (3.2.3), we have $(r-1)(r-1)(r-1) = 0$, $r = 1, 1, 1$
Thus, $\rho(1) = 0 = \rho'(1)$, $\rho''|1| = 0 = 2!\,\delta|-1|$
$\sum_{j=0}^{k} \alpha j = 1 - 3 + 3 - 1 = 0$

Therefore the method (2.1.12) is consistent, convergent and zero stable.

3.2.0 STABILITY PROPERTIES OF THE METHOD

A linear multistep method of the form (1.0.2) is said to be

(i) Zero stable if no root of the first characteristics polynomial has modulus greater than one that is it must be within a unit circle.

(ii) Absolutely or relatively stable in a region R of the complex plane if for all $\bar{h} \in$ R; all roots (r_s) of the stability polynomial $\pi(r, \bar{h})$ associated with the method satisfy.

$| r_s | < |$; s = 1, 2, …. K, $\forall$ $|r_s| < |r_1|$

S = 2, 3, 4….., k, [according to [11], [5]] (3.2.1)

REGION OF ABSOLUTE STABILITY

The method in equation (2.1.12) satisfies the conditions for zero stability since there is no root of its first characteristic polynomial ρ(r) that is greater than one since

$\rho(r) = r^3 - 3r^2 + 3r - 1 = 0$ 3.4.2

$(r-1)(r-1)(r-1) = 0$, r = 1, 1, 1

To test for region of absolute stability; the following steps can be followed;

$$\bar{h}(r) = \frac{T(r)}{H(r)} = \frac{8(r^3 - 3r^2 + 3r - 1)}{r^3 + 3r^2 + 3r + 1} \quad 3.4.3$$

Let $r = e^{i\theta} = \cos\theta + i\sin\theta$ 3.4.4

$T(r) = [32\cos^3\theta - 48\cos^2\theta - 4] + i\,[6\sin\theta\cos\theta + 3\sin\theta]$

$H(r) = [4\cos^3\theta + 6\cos^2 - 2] + i\,[6\sin\theta\cos\theta + 3\sin\theta]$ 3.4.5

$$\bar{h}(\theta) = X(\theta) + iY(\theta) \quad 3.4.6$$

$$X(\theta) = \frac{(32\cos^3\theta - 48\cos^2\theta - 4)(4\cos^3\theta + 6\cos^2\theta - 2) + (6\sin\theta\cos\theta + 3\sin\theta)^2}{(4\cos^3\theta + 6\cos^2\theta - 2)^2 + (6\sin\theta\cos\theta + 3\sin\theta)^2}$$

$$Y(\theta) = \frac{(6\sin\theta\cos\theta + 3\sin\theta)(-28\cos^3\theta + 53\cos^2\theta + 2)}{(4\cos^3\theta + 6\cos^2\theta - 2)^2 + (6\sin\theta\cos\theta + 3\sin\theta)^2}$$

where X(θ) was evaluated for values of θ ranges between 0^0 and 180^0 and the results ere tabulated below

Table 2

Θ	0	30	60	90	120	150	180
X(θ)	-1.25	-12.179	-0.995	-1.55	-45	-35.46	-∞

That is, its region of absolute stability is within interval (-45, -∞). It is P-Stable.

4.0 NUMERICAL EXPERIMENTS

4.1 SAMPLED PROBLEMS

$y^{11} = -y$

$Y^1(0) = 1$, $y(0) = 0$

Theoretical Solution

y(x) = Sin X

For implementation of the developed method in solving the sampled problem, FORTRAN programs were developed at step size h = 0.1, 0.01 and 0.001 and computerized for steps k = 3. The performance of the method on the sampled problem was shown below in tabular and graphical forms.

PERFORMANCE OF PROPOSED METHOD ON SAMPLE PROBLEM

Where Step k = 3 and h = 0.1

MESH SIZE (X)	EXACT SOLUTION (ES)	NUMERICAL RESULT (NR)	ERROR (E)
30	0.50006	0.50459	0.00453
60	0.86609	0.86870	0.00260
90	1.00000	0.99998	-0.00002
120	0.86589	0.86326	-0.00263
150	0.49971	0.49516	-0.00454
180	-0.00041	-0.00564	-0.00524
210	-0.50041	-0.50494	-0.00453
240	-0.86630	-0.86890	-0.00260
270	-1.00000	-0.99998	0.00002
300	-0.86569	-0.86305	0.00264
330	-0.49935	-0.49481	0.00455
360	0.00081	0.00605	0.00524
390	0.50076	0.50529	0.00452
420	0.86650	0.86910	0.00260
450	1.00000	0.99998	-0.00002
480	0.86548	0.86284	-0.00264
510	0.49900	0.49445	-0.00455
540	-0.00122	-0.00646	-0.00524
570	-0.50112	-0.50564	-0.00452
600	-0.86670	-0.86930	-0.00260

TABLE 2B: PERFORMANCE OF PROPOSED METHOD ON THE SAMPLE PROBLEM

Where Step k = 3 and h = 0.01

MESH SIZE (X)	EXACT SOLUTION (ES)	NUMERICAL RESULT (NR)	ERROR (E)
30	0.50006	0.50051	0.00045
60	0.86609	0.86635	0.00026
90	1.00000	1.00000	0.00000
120	0.86589	0.86563	-0.00026
150	0.49971	0.49925	-0.00045
180	-0.00041	-0.00093	-0.00052
210	-0.50041	-0.50086	-0.00045
240	-0.86630	-0.86656	-0.00026
270	-1.00000	-1.00000	0.00000
300	-0.86569	-0.86542	0.00026
330	-0.49935	-0.49890	0.00045
360	0.00081	0.00134	0.00052
390	0.50076	0.50122	0.00045
420	0.86650	0.86676	0.00026
450	1.00000	1.00000	0.00000
480	0.86548	0.86522	-0.00026
510	0.49900	0.49855	-0.00045
540	-0.00122	-0.00175	-0.00052
570	-0.50112	-0.50157	-0.00045
600	-0.86670	-0.86696	-0.00026

TABLE 2C: PERFORMANCE OF PROPOSED METHOD ON THE SAMPLE PROBLEM

Where Step k = 3 and h = 0.001

MESH SIZE (X)	EXACT SOLUTION (ES)	NUMERICAL RESULT (NR)	ERROR (E)
30	0.50006	0.50010	0.00005
60	0.86609	0.86612	0.00003
90	1.00000	1.00000	0.00000
120	0.86589	0.86586	-0.00003
150	0.49971	0.49966	-0.00005
180	-0.00041	-0.00046	-0.00005
210	-0.50041	-0.50046	-0.00005
240	-0.86630	-0.86632	-0.00003
270	-1.00000	-1.00000	0.00000
300	-0.86569	-0.86566	0.00003
330	-0.49935	-0.49931	0.00005
360	0.00081	0.00087	0.00005
390	0.50076	0.50081	0.00005
420	0.86650	0.86653	0.00003
450	1.00000	1.00000	0.00000
480	0.86548	0.86546	-0.00003
510	0.49900	0.49895	-0.00005
540	-0.00122	-0.00127	-0.00005
570	-0.50112	-0.50116	-0.00005
600	-0.86670	-0.86673	-0.00003

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FIGURE 2A: GRAPH SHOWING EXACT SOLUTION, NUMERICAL RESULT OF THE SAMPLE PROBLEM where k = 3 and h = 0.1

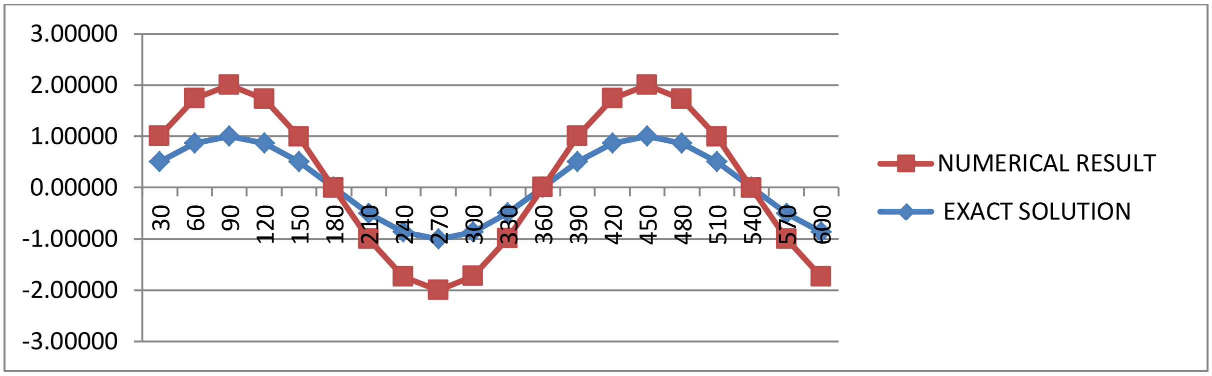


FIGURE 2B: GRAPH SHOWING EXACT SOLUTION, NUMERICAL RESULT OF PROBLEM 1 where k = 3 and h = 0.01

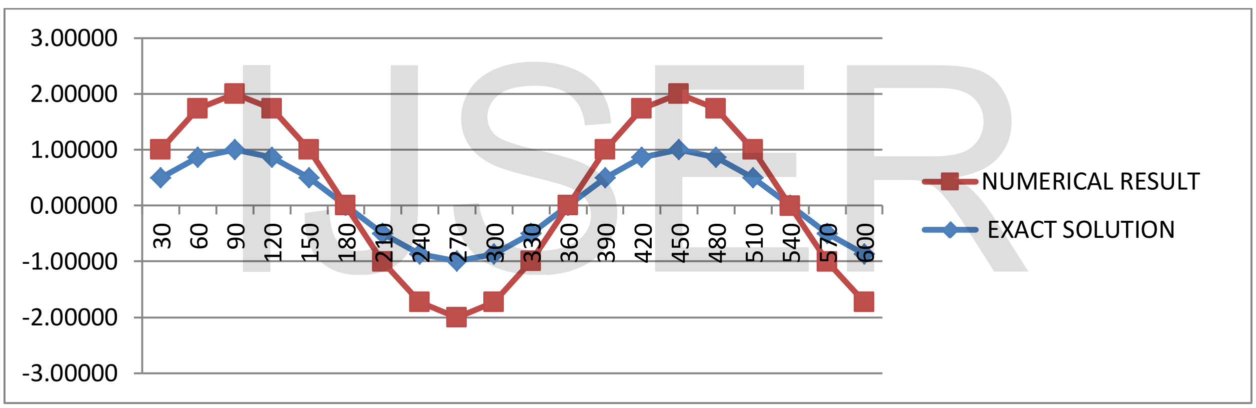
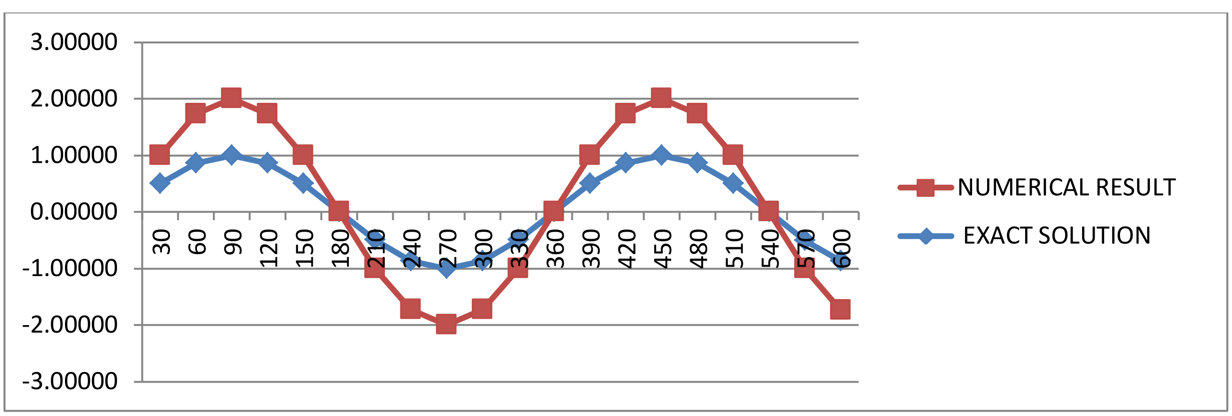


FIGURE 2C: GRAPH SHOWING EXACT SOLUTION, NUMERICAL RESULT OF PROBLEM 1 where k = 3 and h = 0.001



1.5.0 CONCLUSION

In this work, numerical method for solution of periodic initial value problems of second order ordinary differential equation had been discussed. The developed method was analyzed and found to be consistent, convergent and stable.
The performance of the method was implemented on some sampled problems of second order ordinary differential equation with oscillatory solutions. The results show that as the values of h decreases from 0.1 to 0.001, the truncation error approaches zero. It was also observed that as h decreases the graph of the numerical result (NR) and the exact solution (ES) of the proposed methods at each steps of k were nearly overlaps each other and the discretization error vanishes as h tend to zero, showing that the proposed method is accurate and can compare favourably with the result on table 2.

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